An Expoenential-Linear Calculus for Self-Consistent Aggregation

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Abstract

This paper establishes a unified framework for continuous and discrete averaging by interpolating between arithmetic and geometric regimes through a tunable parameter φ. We define a continuous linear calculus where the standard derivative and the exponential derivative đ are viewed as boundary cases of a generalized linear operator. By preserving the fundamental symmetry between a function and its derivative—specifically that the moving average of an accumulated state is equivalent to the integral of the arithmetic average of its underlying rate—we derive a unique "Linear Mean" operator.

Central to this work is the rigorous restoration of the function-centric identity. We utilize a homeomorphism lemma to prove that the linear mean is coordinate-independent and defined purely by local window values, effectively removing the dependence on an arbitrary fixed point x_0. This continuous foundation is then discretized to produce the **Unified Discrete Linear Mean** $\tilde{S}_\varphi$ for a general set S:

$$\tilde{S}_\varphi = \frac{\prod_{x\in S}(x+1-\varphi)^{\frac{1}{|S|}}}{\varphi} + \frac{\varphi-1}{\varphi}$$

We further demonstrate that this mean satisfies the property of associativity (decomposability), allowing for hierarchical data aggregation within a linearized state space. By bridging the gap between additive and multiplicative averaging, this framework provides a robust statistical tool for analyzing systems in transition between linear and exponential growth models.

1 Foundations of Exponential Calculus

A calculus that aggregates exponentially.

Imagine, when the derivative of e^x is 1, because in this equation:

$$e^x = \lim_{i\to 0} \prod_{n=0}^{x/i} (a+i);$$

a equals to 1.

1.1 The Exponential Differential and Derivative

We define the exponential differential đy as the relative change in y. Given an initial value y_0 and a subsequent value y_1:

$$đy = \frac{y_1}{y_0} - 1$$

As $\Delta x \to 0$, $y_1 \to y_0$, ensuring đ$y \to 0$.

1.1.1 Geometric Interpretation

Considering the curve $f(x)$, we visualize the differential change over an interval Δx:

$$\begin{aligned} đf(x) &= \frac{f(x+\Delta x)}{f(x)} - 1 = \frac{f(x) + f'(x)\Delta x}{f(x)} - 1 \\ &= \frac{f'(x)\Delta x}{f(x)} \end{aligned}$$

Dividing by the change in x yields the **Exponential Derivative**:

$$\frac{đf(x)}{dx} = \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{f(x)} - 1}{\Delta x} = \frac{f'(x)}{f(x)} \tag{1}$$

1.2 Inverse-Exponential Operations

While the exponential derivative evaluates the ratio of y, the inverse-exponential derivative focuses on the ratio of x.

1.2.1 The Inverse Exponential Derivative

$$\begin{aligned} \frac{dy}{đx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\frac{x+\Delta x}{x} - 1} \\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot x = f'(x) \cdot x \end{aligned}$$

Example: For $f(x) = \ln x$, $\frac{d(\ln x)}{đx} = \frac{1}{x} \cdot x = 1$.

1.2.2 Inverse Exponential Integration

The inverse exponential integral recovers the function from its x-relative derivative:

$$\int g(x) đx = \int \frac{g(x)}{x}\, dx \tag{2}$$

Proof: If $\frac{dy}{dx} \cdot x = g(x)$, then $y' = \frac{g(x)}{x}$, which integrates to $\int \frac{g(x)}{x} dx$.

2 Exponential Integration

2.1 Discrete Definition

The exponential integral $⨎_a^b$ represents the limit of an infinite product of infinitesimal growth steps. For $i \to 0$:

$$F(x) = ⨎_a^b f(x)\, dx = \lim_{i \to 0} \prod_{n=0}^{\frac{b-a}{i}} [1 + f(x_n) \cdot i] \tag{3}$$

where $x_n = a + ni$.

2.2 Fundamental Relationship with Standard Calculus

To solve for the continuous form of the exponential integral, we treat the derivative relationship as a separable differential equation:

$$\begin{gathered}
\frac{\text{đ}y}{dx} = g(x) \implies \frac{1}{y}\frac{dy}{dx} = g(x) \\
\int \frac{1}{y} dy = \int g(x)\, dx \\
\ln|y| = \int g(x)\, dx + C \\
y = Ce^{\int g(x)\, dx}
\end{gathered}$$

For the definite case, this resolves to the **Exponential Identity**:

$$⨎_a^b g(x)\, dx = e^{\int_a^b g(x)\, dx} \tag{4}$$

2.3 Example: The Unit Identity

When $g(x) = 1$, the accumulation over $[0, x]$ recovers the natural exponential function:

$$⨎_0^x 1\, dx = e^{\int_0^x 1\, dx} = e^x$$

Exponential Definitions

1. Definition of e

In this framework, e is defined by the infinite product of infinitesimal growth steps:

$$e = \lim_{\Delta x \to 0} \prod_{n=0}^{\cdot \frac{1}{\Delta x}} (1 + \Delta x)$$

2. Definition of e^x

The function e^x is defined based on the sign of x:

$$e^x = \begin{cases} \displaystyle\lim_{\Delta x \to 0^+} \prod_{n=0}^{x/\Delta x} (1 + \Delta x) & (x \geq 0) \\ \displaystyle\lim_{\Delta x \to 0^-} \frac{1}{\prod_{n=0}^{-x/\Delta x} (1 + \Delta x)} & (x < 0) \end{cases}$$

3. Scaling the Exponent: e^{kx}

For a constant k, the function scales its internal growth rate:

$$e^{kx} = \begin{cases} \displaystyle\lim_{\Delta x \to 0^+} \prod_{n=0}^{\frac{kx}{\Delta x}} (1 + \frac{\Delta x}{k} \cdot k) & (x \geq 0) \\ \displaystyle\lim_{\Delta x \to 0^-} \frac{1}{\prod_{n=0}^{-\frac{kx}{\Delta x}} (1 + \frac{\Delta x}{k} \cdot k)} & (x < 0) \end{cases}$$

4. Derivation of $đe^{kx}$

We evaluate the exponential differential of e^{kx} by observing the ratio of the product sequences:

$$
\begin{aligned}
e^{kx} &= \lim_{\Delta x \to 0} \prod_{n=0}^{\frac{x}{\Delta a}} (1 + \Delta x), \quad \text{where } \Delta a = \frac{\Delta x}{k} \\
&= \lim_{\Delta x \to 0} \prod_{n=0}^{\frac{x}{\Delta a}} (1 + k \cdot \Delta a)
\end{aligned}
$$

Calculating the differential:

$$
\begin{aligned}
đe^{kx} &= \frac{\prod_{n=0}^{\frac{x+\Delta a}{\Delta a}} (1 + k \cdot \Delta a)}{\prod_{n=0}^{\frac{x}{\Delta a}} (1 + k \cdot \Delta a)} - 1 \\
&= (1 + k \cdot \Delta a) - 1 \\
&= k \cdot \Delta a
\end{aligned}
$$

3 Advanced Exponential Relations

3.1 The Exponential Chain Rule

We establish the link between linear and exponential change by observing the growth of a constant rate k. If $y = e^{kx}$:

$$
\frac{e^{kx+\Delta x}}{e^{kx}} - 1 = e^{\Delta x} - 1 \approx k \cdot \Delta x
$$

For a general function $f(x)$, let k be the exponential derivative $k = \frac{đf(x)}{dx}$. The relationship between the linear change (dx) and exponential change ($đf(x)$) is given by:

$$
\frac{đf(x)}{dx} \cdot dx = đf(x) \tag{5}
$$

This implies the fundamental approximation identity:

$$
e^{\frac{đf(x)}{dx} \cdot dx} \approx 1 + đf(x) \tag{6}
$$

3.2 Differential Identities

The exponential derivative $\frac{đy}{dx}$ represents the percentage increase in y during an interval dx. This leads to several useful relationships between differentials:

- **Direct Relation:** $\frac{đy}{dx} = \frac{dy/y}{dx} \implies đy = \frac{dy}{y}$
- **Function Update:** $f(x + dx) = f(x)(1 + đy)$
- **Exponential Update:** $f(x + dx) = f(x)e^{đy}$

4 Definite Exponential Integration

The operation $⨏ f(x)\, dx$ computes the compounding of the exponential differential over a range.

4.1 Multiplicative Property Derivation

Using the relationship $đy = f(x)\,dx$, we can restore a function's value by integrating the differential from an initial state (x_0, y_0) to (x, y):

$$\fint_{y_0}^{y} đy = \fint_{x_0}^{x} f(x)\,dx$$

Since the left side represents total compound growth (y/y_0):

$$\begin{aligned} \frac{y}{y_0} &= e^{\int_{x_0}^{x} f(x)\,dx} \\ y &= y_0 e^{\int_{x_0}^{x} f(x)\,dx} \end{aligned}$$

4.2 Discrete Product-Integral Limit

Formally, we define the definite integral as the limit of an infinite product of growth factors:

$$\fint_{a}^{b} f(x)\,dx = \lim_{n\to\infty} \prod_{i=1}^{n} \left(1 + \frac{b-a}{n} f\left(x_i\right)\right)$$

Applying the identity $1 + \epsilon \approx e^{\epsilon}$ for infinitesimal ϵ:

$$\begin{aligned} &= \lim_{n\to\infty} \prod_{i=1}^{n} \left(e^{\frac{b-a}{n} f(x_i)}\right) \\ &= e^{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(x_i)} = e^{\int_a^b f(x)\,dx} \end{aligned}$$

5 Linear Differentiation

5.1 Motivation: Unifying Two Orders of Change

We have established two distinct modes of differentiation based on how the differential dy relates to the current value of y:

- **0-Order (Standard):** $dy = f(x)\,dx$. Here, the change is independent of the current state y.
- **1-Order (Exponential):** $dy = f(x)y\,dx$. Here, the change is proportional to y, implying that as $y \to 0$, $dy \to 0$.

Our goal is to combine these behaviors into a unified **Linear Differentiation** framework.

5.2 Derivation and Re-parameterization

To account for both behaviors, we express the total differential dy as a linear combination using constants $a, b \in \mathbb{R}$:

$$dy = af(x)y\,dx + bf(x)\,dx$$

Recognizing that dy is invariant under scaling of the system, we re-parameterize the weights a and b using a scaling factor C and a distribution parameter φ:

$$dy = C\varphi f(x)y\,dx + C(1-\varphi)f(x)\,dx$$
$$\text{where: } C = a+b, \quad \varphi = \frac{a}{a+b}, \quad \varphi \in [0,1]$$

By absorbing the global scaling factor C into the function $f(x)$, we arrive at the **Final Unified Form**:

$$dy = [\varphi y + (1-\varphi)]\,f(x)\,dx \tag{7}$$

5.3 Definition of the Linear Derivative

Let $f(x)$ be defined as the **Linear Derivative**, denoted by $\frac{\overset{\varphi}{đ}y}{dx}$. By partitioning the total differential into its value-dependent and constant components, we obtain the fundamental relation:

$$dy = (\varphi y + 1 - \varphi)\,\frac{\overset{\varphi}{đ}y}{dx}\,dx$$

Isolating the derivative ratio, we define the operator as:

$$\frac{\overset{\varphi}{đ}y}{dx} = \frac{1}{\varphi y + 1 - \varphi}\frac{dy}{dx} \tag{8}$$

5.3.1 Geometric Interpretation

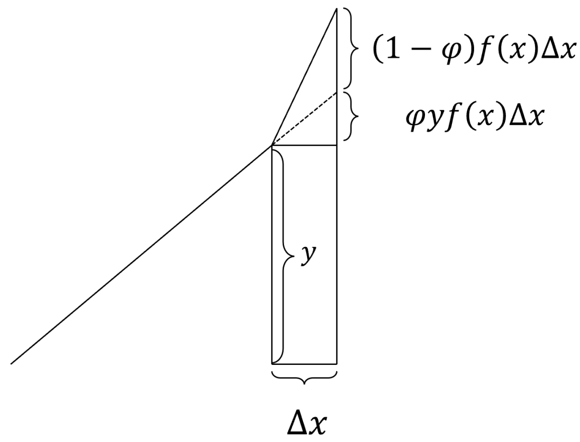

Figure 1: The total change dy is composed of a zeroth-order component, $(1-\varphi)f(x)dx$, and a growth component, $\varphi y f(x)dx$, which scales with the current value of y.

6 Linear Integration

6.1 Definition of the Operator

The linear derivative of y with parameter φ represents the rate of change relative to the weighted base $(\varphi y + 1 - \varphi)$:

$$f(x) = \frac{\text{đ}^{\varphi} y}{dx}$$

This implies the following fundamental differential relationship:

$$dy = (\varphi y + 1 - \varphi)\, f(x)\, dx$$
$$\frac{dy}{\varphi y + 1 - \varphi} = f(x)\, dx$$

6.2 The Inverse Operation

We define the linear integral $F(x) = \displaystyle\fint^{\varphi} f(x)\, dx$ as the operator satisfying $\frac{\text{đ}^{\varphi} F(x)}{dx} = f(x)$.

6.2.1 Solving the System

To find the explicit form of y, we integrate the differential relationship:

$$\int \frac{dy}{\varphi y + 1 - \varphi} = \int f(x)\, dx$$

Applying the logarithmic integral (with $du = \varphi\, dy$):

$$\frac{1}{\varphi} \ln|\varphi y + 1 - \varphi| = \int f(x)\, dx + C$$
$$\ln|\varphi y + 1 - \varphi| = \varphi \int f(x)\, dx + \varphi C$$
$$\varphi y + 1 - \varphi = e^{\varphi \int f(x)\, dx + \varphi C}$$

Letting the unified constant be $C' = e^{\varphi C}$, we obtain the general form:

$$y = \frac{C' e^{\varphi \int f(x)\,dx} - (1-\varphi)}{\varphi} \tag{9}$$

7 Limit Analysis of the Linear Integral

The stability of the linear framework is demonstrated by its convergence to established calculus systems at its boundaries.

7.1 The Multiplicative Limit ($\varphi \to 1$)

As $\varphi \to 1$, the system resolves into pure exponential calculus:

$$\begin{aligned}\lim_{\varphi \to 1} y &= \frac{C' e^{1 \cdot \int f(x)\,dx} - (1-1)}{1} \\ &= C' e^{\int f(x)\,dx}\end{aligned}$$

This confirms the identity where exponential accumulation is the exponentiation of the standard integral.

7.2 The Additive Limit ($\varphi \to 0$)

As $\varphi \to 0$, we apply the Taylor expansion $e^u = 1 + u + \dots$ to the exponential term:

$$\begin{aligned}y &= \lim_{\varphi \to 0} \left(\frac{1 + \varphi\left(\int f(x)\,dx + C\right) - 1}{\varphi} + 1 \right) \\ &= \frac{\varphi\left(\int f(x)\,dx + C\right)}{\varphi} + 1 \\ &= \int f(x)\,dx + C'\end{aligned}$$

Result: The framework successfully resolves to standard Newtonian integration. This proves that standard calculus is a special case of the linear system where the value-dependent growth weight is zero.

The Discrete Challenge

The discrete definition of linear integration is not immediately clear. If we derive it from the definition of linear derivatives:

$$\Delta y = \varphi f(x) y \Delta x + (1-\varphi) f(x) \Delta x$$

This implies an iterative accumulation:

$$y_{i+1} = y_i + f(x_i)\left(\varphi y_i + (1-\varphi)\right)\Delta x$$

As noted, the order in which the y value is accumulated can result in different integration results, that is, sometimes the linear integration defined in this way will result in different values over the same curve even with the same starting y_0.

8 Definite Linear Integration

The effect of this path dependency is also shown as we try to take the "Linear" average using this discrete definition; the order in which the elements of a set are "added" to the computation can change the calculated value of the mean, which is not desired.

Thus, I trial to derive the form of the discrete definition backward directly with the continuous form (which disregards the order of accumulation):

$$\sideset{}{^{\varphi}}\fint_a^b f(x)\,dx = \frac{e^{\varphi \int_a^b f(x)\,dx} - 1}{\varphi} + 1$$

Comparison with Regular Definite Integration

Note that, for regular definite integration, we have:

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

where $F(x)$ is the indefinite integral of $f(x)$ with constant C, and C gets cancelled out. Similarly:

$$\fint_a^b f(x)\,dx = \frac{F(b)}{F(a)} = F(x)\ddagger_a^b$$

where $F(x)$ is the indefinite exponential integral of $f(x)$ with constant factor C, and C gets cancelled out. We want to find the same method that allows us to directly compute results from the indefinite linear integral.

8.1 Restoration of the Function y

To determine the behavior of y under linear integration, we evaluate the integral of the linear differential $\text{đ}^{\varphi} y$ from an initial state y_0 to a final state y_1. Recalling that $\text{đ}^{\varphi} y = \frac{dy}{\varphi y + 1 - \varphi}$, we take the linear differentiation of the original function and restore it with linear integration using our proposed definite form:

$$\begin{aligned}
\sideset{}{^{\varphi}}\fint_{y_0}^{y_1} \text{đ}^{\varphi} y &= \frac{e^{\varphi \int_{y_0}^{y_1} \frac{dy}{\varphi y + 1 - \varphi}} - 1}{\varphi} + 1 \\
&= \frac{e^{\varphi\left(\frac{1}{\varphi} \ln|\varphi y + 1 - \varphi|\Big|_{y_0}^{y_1}\right)} - 1}{\varphi} + 1 \\
&= \frac{\frac{\varphi y_1 + 1 - \varphi}{\varphi y_0 + 1 - \varphi} - 1}{\varphi} + 1 \\
&= \frac{\varphi y_1 + 1 - \varphi}{\varphi(\varphi y_0 + 1 - \varphi)} + \frac{\varphi - 1}{\varphi}
\end{aligned}$$

9 The Fundamental Theorem and Linear Aggregation

9.1 Corollaries of the Fundamental Theorem

Corollary 1: The definite linear integral can be computed directly from the indefinite linear integral $F(x)$ using a specific evaluation notation:

$$\fint^{\varphi}{}_{a}^{b} f(x)\,dx = F(x)\Big|_{\varphi,a}^{b} = \frac{\varphi F(b) + 1 - \varphi}{\varphi(\varphi F(a) + 1 - \varphi)} + \frac{\varphi - 1}{\varphi} \tag{10}$$

Corollary 2: Based on the restoration of y derived previously, the final value $y(b)$ is expressed as a linear aggregation of the initial value $y(a)$ and the accumulated integral:

$$y(b) = \varphi\left(y(a)\fint^{\varphi}{}_{a}^{b} f(x)\,dx + 1\right) + (1-\varphi)\left(y(a) + \fint^{\varphi}{}_{a}^{b} f(x)\,dx\right) - 1 \tag{11}$$

This relationship allows us to define a unified binary aggregation operator, $y(a) *^{\varphi} \fint^{\varphi}{}_{a}^{b} f(x)\,dx$.

9.2 The Linear Aggregation Operator

We define the binary operator $*^{\varphi}$ for any $a, b \in \mathbb{R}$:

$$a *^{\varphi} b = \varphi(ab + 1) + (1-\varphi)(a + b) - 1 \tag{12}$$

Fundamental Theorem of Linear Calculus
Let y be a function of x. The change in y over the interval $[a, b]$ is given by the linear integral of its linear derivative:

$$y(b) = y(a) *^{\varphi} \left(\fint^{\varphi}{}_{a}^{b} \frac{đ^{\varphi} y}{dx}\,dx\right) \tag{13}$$

The operator $*^{\varphi}$ satisfies three critical properties:

1. **Commutativity:** $a *^{\varphi} b = b *^{\varphi} a$.
2. **Additive Limit ($\varphi \to 0$):** $a *^{0} b = a + b - 1$.
3. **Multiplicative Limit ($\varphi \to 1$):** $a *^{1} b = ab$.

9.3 The Discrete Definition

Analogous to summation and products, we derive the discrete aggregation scheme from the continuous form:

$$\fint^{\varphi}{}_{a}^{b} f(x)\,dx = \lim_{n\to\infty} \frac{\prod_{i=1}^{n} e^{\varphi f(x_i)\Delta x} - 1}{\varphi} + 1$$

Applying the Taylor expansion $e^{\varphi f \Delta x} \approx 1 + \varphi f \Delta x$ for infinitesimal $\Delta x \to 0$:

$$\fint^{\varphi}{}_{a}^{b} f(x)\,dx = \frac{\prod_{i=1}^{n} (\Delta x f(x_i) + 1)^{\varphi} - 1}{\varphi} + 1$$

10 Foundations of Continuous Averaging

To develop a unified theory of means, we first define the continuous extensions of arithmetic and geometric averaging over a sliding window of length t.

10.1 Arithmetic Mean (Additive)

For a discrete set $S \subseteq \mathbb{R}$, the arithmetic mean $\bar{S}$ is defined by the sum of its elements divided by the set cardinality:

$$\bar{S} = \frac{\sum_{x \in S} x}{|S|}$$

We extend this to a function $f : \mathbb{R} \to \mathbb{R}$ over the interval $[x - t, x]$ by discretizing the range into $|N|$ elements. Let $x_i = x - t + \frac{it}{|N|}$ for $i \in \{1, \ldots, |N|\}$. The moving arithmetic mean $\bar{f}^t(x)$ is derived as follows:

$$\begin{aligned}
\bar{f}^t(x) &= \frac{\sum_{i=1}^{|N|} f(x_i)}{|N|} \\
&= \sum_{i=1}^{|N|} \frac{1}{|N|} f\left(x - t + \frac{it}{|N|}\right) \\
&= \frac{1}{t} \sum_{i=1}^{|N|} \frac{t}{|N|} f\left(x - t + \frac{it}{|N|}\right)
\end{aligned}$$

In the limit $|N| \to \infty$, the summation converges to a definite integral:

$$\bar{f}^t(x) = \frac{1}{t} \int_{x-t}^{x} f(x)\, dx$$

10.2 Geometric Mean (Multiplicative)

For a given set S where $\forall x \in S, x > 0$, the geometric mean $\hat{S}$ is defined by the $|S|$-th root of the product of its elements:

$$\hat{S} = \left(\prod_{x \in S} x\right)^{\frac{1}{|S|}}$$

Extending this to a continuous definition over the range $t \in \mathbb{R}$ (assuming $f(x) > 0$), we evaluate the product limit:

$$\hat{f}^t(x) = \left(\prod_{i=1}^{|N|} f(x_i)\right)^{\frac{1}{|N|}}$$

Applying the property $f(x_i) = e^{\ln(f(x_i))}$:

$$\begin{aligned}
&= \left(\prod_{i=1}^{|N|} e^{\ln(f(x_i))}\right)^{\frac{1}{|N|}} \\
&= e^{\frac{1}{|N|} \sum_{i=1}^{|N|} \ln(f(x_i))}
\end{aligned}$$

In the continuous limit $|N| \to \infty$, we establish the equivalence with the exponential integral:

$$\hat{f}^t(x) = \exp\left(\frac{1}{t}\int_{x-t}^{x} \ln(f(x))\,dx\right) = \left(\displaystyle\fint_{x-t}^{x} \ln(f(x))\,dx\right)^{\frac{1}{t}}$$

Note: If $f(x) \leq 0$ for any part of the interval, the geometric mean $\hat{f}^t(x)$ is generally defined as 0 or is undefined in the real number space.

11 Symmetry of Nested Means

11.1 Calculus Theorem for the Nested Arithmetic Mean

Let $F : \mathbb{R} \to \mathbb{R}$ be a function with derivative $f : \mathbb{R} \to \mathbb{R}$. Without loss of generality, assume $F(x) = \int_{x_0}^{x} f(s)\,ds + y_0$. The moving arithmetic average $\bar{F}^t(x)$ is derived by integrating the accumulated state:

$$\begin{aligned}
\bar{F}^t(x) &= \frac{\int_{x-t}^{x}\left(\int_{x_0}^{x} f(s)\,ds + y_0\right) ds}{t} \\
&= \int_{x-t}^{x}\int_{x_0}^{x} \frac{f(s)}{t}\,ds^2 + \int_{x-t}^{x} \frac{y_0}{t}\,ds \\
&= \int_{x_0}^{x}\int_{x-t}^{x} \frac{f(s)}{t}\,ds^2 + \frac{(x-(x-t))}{t}y_0 \\
&= \int_{x_0}^{x} \bar{f}^t(s)\,ds + y_0
\end{aligned}$$

Result: The accumulated average of a derivative is equivalent to the change in the average of the function itself: $\frac{d}{dx}\bar{F}^t(x) = \bar{f}^t(x)$.

11.2 Calculus Theorem for the Nested Geometric Mean

For functions $F : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, where $F(x)$ is the exponential accumulation of $f(x)$:

$$F(x) = y_0 \fint_{x_0}^{x} f(x)\,dx \implies \frac{\text{đ}F(x)}{dx} = f(x)$$

We define the moving geometric mean $\hat{F}^t(x)$ over window t as the exponentiated log-average:

$$\begin{aligned}
\hat{F}^t(x) &= \left(\fint_{x-t}^{x} \ln(F(x))\,dx\right)^{\frac{1}{t}} \\
&= \left(\fint_{x-t}^{x} \ln\left(y_0 e^{\int_{x_0}^{x} f(s)\,ds}\right)\,dx\right)^{\frac{1}{t}} \\
&= \left(\fint_{x-t}^{x} \left(\int_{x_0}^{x} f(s)\,ds + \ln(y_0)\right)dx\right)^{\frac{1}{t}}
\end{aligned}$$

Applying the addition rule for exponential integration:

$$
\begin{aligned}
\hat{F}^t(x) &= e^{\ln(y_0)} \cdot e^{\frac{1}{t}\int_{x-t}^{x}\int_{x_0}^{x} f(s)\,ds\,dx} \\
&= y_0 \exp \int_{x_0}^{x} \left(\frac{\int_{s-t}^{s} f(u)\,du}{t} \right) ds \\
&= y_0 \fint_{x_0}^{x} \bar{f}^t(x)\,dx
\end{aligned}
$$

Theorem 1 (Calculus Theorem for Nested Geometric Average)**.** *The geometric average of an exponential function is equivalent to the exponential integral of the arithmetic average of its rate of change:*

$$
\hat{F}^t(x) = y_0 \fint_{x_0}^{x} \bar{f}^t(x)\,dx
$$

Corollary 1. *Taking the exponential derivative of the geometric average yields the arithmetic average of the growth rate:*

$$
\frac{\text{đ}\hat{F}^t(x)}{dx} = \bar{f}^t(x)
$$

11.2.1 Verification: The Limit $t \to 0$

As the averaging window vanishes, the geometric average resolves to the instantaneous point value:

$$
\begin{aligned}
\hat{F}^0(x) &= y_0 \fint_{x_0}^{x} \bar{f}^0(x)\,dx \\
&= y_0 \fint_{x_0}^{x} f(x)\,dx = F(x)
\end{aligned}
$$

12 Linear Averaging: The Interpolated Mean

Each method of integration possesses an associated discrete definition of an average; consequently, for linear integration, we seek a corresponding definition for taking means. The linear integral acts as a structural bridge between established systems:

- **Additive Boundary** ($\varphi = 0$)**:** $\fint^{b}_{{}^{0}a} f(x)\,dx = \int_a^b f(x)\,dx$
- **Multiplicative Boundary** ($\varphi = 1$)**:** $\fint^{b}_{{}^{1}a} f(x)\,dx = \fint_a^b f(x)\,dx$

The resulting linear mean must be an interpolation between the arithmetic and geometric mean, derived by preserving the fundamental symmetry observed in standard and exponential systems.

12.1 Preserving Symmetry and Integral Representation

We observe that for both boundary cases, the moving average $\bar{F}^t(x)$ satisfies a specific derivative property and a corresponding integral representation:

- **Arithmetic Symmetry** ($\varphi = 0$)**:**

$$
\frac{d}{dx}\bar{F}^t(x) = \bar{f}^t(x) \iff \bar{F}^t(x) = y_0 + \int_{x_0}^{x} \bar{f}^t(s)\,ds
$$

- **Geometric Symmetry ($\varphi = 1$):**

$$\frac{\text{đ}\hat{F}^t(x)}{dx} = \bar{f}^t(x) \iff \hat{F}^t(x) = y_0 \cdot \fint_{x_0}^{x} \bar{f}^t(s)\, ds$$

To extend this property to the **Linear Mean** $\tilde{F}^t_\varphi(x)$, we define the operator such that it maintains this derivative symmetry:

$$\frac{\text{đ}^\varphi \tilde{F}^t_\varphi(x)}{dx} = \bar{f}^t(x), \text{ where } \frac{\text{đ}^\varphi F}{dx} = f \tag{14}$$

12.2 Construction of the Unified Linear Mean

The integral forms above naturally generalize to the linear framework via the aggregation operator $*^\varphi$. We first recall the restoration identity for the base function $F(x)$:

$$F(x) = F_0 *^\varphi \fint^{\varphi\, x}_{x_0} f(x)\, dx \tag{15}$$

To satisfy the consistency requirement that the moving average recovers the function as the window vanishes ($\lim_{t\to 0} \tilde{F}^t_\varphi(x) = F(x)$), we propose the following construction for the **Moving Linear Mean**:

$$\tilde{F}^t_\varphi(x) = F_0 *^\varphi \fint^{\varphi\, x}_{x_0} \bar{f}^t(s)\, ds \tag{16}$$

Theorem 2 (Calculus Theorem for Nested Linear Mean)**.** *Assuming initial conditions $F_0 = C_0$, the linear moving average of a function is given by the linear aggregation of its initial state and the linear integral of its arithmetic-averaged rate:*

$$\tilde{F}^t_\varphi(x) = F_0 *^\varphi \left(\frac{e^{\varphi \int_{x_0}^{x} \bar{f}^t(s)\, ds} - 1}{\varphi} + 1 \right)$$

12.3 Verification of the Boundaries

Verification: The Additive Limit ($\varphi \to 0$)

As φ vanishes, the linear mean should resolve to the standard arithmetic mean $\bar{F}^t(x)$:

$$\begin{aligned} \lim_{\varphi\to 0} \tilde{F}^t_\varphi(x) &= F_0 + \int_{x_0}^{x} \bar{f}^t(x)\, dx + 1 - 1 \\ &= F(x) - F(x_0) + F(x_0) \\ &= \int_{x_0}^{x} \bar{f}^t(x)\, dx + F(x_0) = \bar{F}^t(x) \end{aligned}$$

Result: $\tilde{F}^t_0(x) = \bar{F}^t(x)$, confirming consistency with the Calculus Theorem for arithmetic means.

Verification: The Multiplicative Limit ($\varphi \to 1$)

Similarly, we test the linear mean as it approaches the multiplicative boundary:

$$\begin{aligned} \tilde{F}^t_1(x) &= F_0 *^1 \fint^{1\, x}_{x_0} \bar{f}^t(s)\, ds \\ &= F_0 \cdot \fint_{x_0}^{x} \bar{f}^t(s)\, ds \\ &= \hat{F}^t(x) \end{aligned}$$

Result: This matches the **Calculus Theorem for Geometric Mean** derived on Page 13. Hence, $\tilde{F}^t_\varphi(x)$ is a robust interpolation between the arithmetic mean $\bar{F}^t(x)$ and the geometric mean $\hat{F}^t(x)$.

12.4 Recovering from the nested Mean

Using the definition of the linear integral, we can express the continuous moving linear average as:

$$\begin{aligned}\tilde{F}^t_\varphi(x) &= F_0 *^\varphi {}^{\varphi}\!\!\fint_{x_0}^{x} \bar{f}^t(s)\, ds \\ &= F_0 *^\varphi {}^{\varphi}\!\!\fint_{x_0}^{x} \left(\frac{F(s) - F(s-t)}{t}\right) ds\end{aligned}$$

12.4.1 Defining the Kernel Components

We begin with the linear mean expressed as an aggregation of the initial state F_0 and the averaged rate:

$$\tilde{F}^t_\varphi(x) = {}^{\varphi}\!\!\fint_{x-t}^{x} \left(g + \frac{1}{t}\int_{x_0}^{x} f(s)\, ds\right) dx$$

To solve for the constant term g, we observe that the initial state F_0 is represented as a linear integral over the window t:

$$\begin{aligned}{}^{\varphi}\!\!\fint_{x-t}^{x} g\, dx = F_0 &\implies \frac{e^{\varphi g t} - 1}{\varphi} + 1 = F_0 \\ &\implies e^{\varphi g t} = (F_0 - 1)\varphi + 1 \\ &\implies g = \frac{\ln((F_0 - 1)\varphi + 1)}{t\varphi}\end{aligned}$$

Next, we isolate the logarithmic form of the accumulated standard integral from the linear integral identity:

$$\begin{aligned}{}^{\varphi}\!\!\fint_{x_0}^{x} f(s)\, ds &= \frac{e^{\varphi \int_{x_0}^{x} f(s)\, ds} - 1}{\varphi} + 1 \\ \implies \int_{x_0}^{x} f(s)\, ds &= \frac{\ln\left(\varphi\left({}^{\varphi}\!\!\fint_{x_0}^{x} f(s)\, ds - 1\right) + 1\right)}{\varphi}\end{aligned}$$

12.4.2 Algebraic Merger and Substitution

Substituting these into the integrand, we combine the terms under a common denominator $t\varphi$:

$$\tilde{F}^t_\varphi(x) = {}^{\varphi}\!\!\fint_{x-t}^{x} \left(\frac{\ln((F_0 - 1)\varphi + 1)}{t\varphi} + \frac{\ln\left(\varphi\left({}^{\varphi}\!\!\fint_{x_0}^{x} f(s)\, ds - 1\right) + 1\right)}{t\varphi}\right) dx$$

Using the logarithmic identity $\ln(A) + \ln(B) = \ln(AB)$:

$$= {}^{\varphi}\!\!\fint_{x-t}^{x} \left(\frac{\ln\left([(F_0 - 1)\varphi + 1] \cdot \left[\varphi\left({}^{\varphi}\!\!\fint_{x_0}^{x} f(s)\, ds - 1\right) + 1\right]\right)}{t\varphi}\right) dx$$

12.4.3 Final Cancellation and Function-Centric Form

Recalling the evaluation of the linear integral $\displaystyle\fint_{x_0}^{x} f(x)\,dx = \frac{\varphi F(x)+1-\varphi}{\varphi(\varphi F_0+1-\varphi)} + \frac{\varphi-1}{\varphi}$, we substitute this into our product:

$$\begin{aligned}\tilde{F}_\varphi^t(x) &= \fint_{x-t}^{x} \left(\frac{\ln\left((F_0\varphi - \varphi + 1)\left(\frac{\varphi F(x)+1-\varphi}{F_0\varphi+1-\varphi}\right)\right)}{t\varphi} \right) dx \\ &= \fint_{x-t}^{x} \left(\frac{\ln(\varphi F(x) + 1 - \varphi)}{t\varphi} \right) dx\end{aligned}$$

Theorem 3 (Linear Mean Identity)**.** *The moving linear average of a function is independent of the initial evaluation point x_0 and is defined purely by the values of the function F within the averaging window:*

$$\tilde{F}_\varphi^t(x) = \fint_{x-t}^{x} \frac{\ln(\varphi F(x) + 1 - \varphi)}{t\varphi}\, dx \tag{17}$$

13 A Homeomorphic Property

Lemma 1 (Addition Rule for Linear Integration)**.** *The linear integral of a sum of functions is equivalent to the linear aggregation ($*^\varphi$) of their individual linear integrals:*

$$\fint_a^b (g(x) + f(x))\,dx = \left(\fint_a^b g(x)\,dx\right) *^\varphi \left(\fint_a^b f(x)\,dx\right)$$

Proof. We begin by evaluating the left-hand side using the definite form of the linear integral:

$$\begin{aligned}\fint_a^b (g(x) + f(x))\,dx &= \frac{e^{\varphi \int_a^b (g(x)+f(x))\,dx} - 1}{\varphi} + 1 \\ &= \frac{e^{\varphi \int_a^b g(x)\,dx} \cdot e^{\varphi \int_a^b f(x)\,dx} - 1}{\varphi} + 1\end{aligned}$$

Now, we evaluate the right-hand side. Let $A = \fint_a^b g(x)\,dx$ and $B = \fint_a^b f(x)\,dx$. Recalling the definition of the linear aggregation operator $*^\varphi$:

$$A *^\varphi B = \varphi(AB + 1) + (1 - \varphi)(a + b) - 1$$

Substituting the explicit integral forms for A and B:

$$\begin{aligned}A *^\varphi B = \varphi &\left(\left(\frac{e^{\varphi \int_a^b g(x)\,dx} - 1}{\varphi} + 1\right)\left(\frac{e^{\varphi \int_a^b f(x)\,dx} - 1}{\varphi} + 1\right) + 1\right) \\ &+ (1 - \varphi)\left(\left(\frac{e^{\varphi \int_a^b g(x)\,dx} - 1}{\varphi} + 1\right) + \left(\frac{e^{\varphi \int_a^b f(x)\,dx} - 1}{\varphi} + 1\right)\right) - 1\end{aligned}$$

Upon expansion, the linear terms and constants cancel systematically:

$$A *^\varphi B = \varphi \left(\frac{(e^{\varphi \int g} - 1 + \varphi)(e^{\varphi \int f} - 1 + \varphi)}{\varphi^2} + 1 \right) + (1 - \varphi) \left(\frac{e^{\varphi \int g} + e^{\varphi \int f} - 2 + 2\varphi}{\varphi} \right) - 1$$

Simplifying the algebraic structure leads back to the unified exponential form:

$$= \frac{e^{\varphi \int_a^b g(x)\,dx} \cdot e^{\varphi \int_a^b f(x)\,dx} - 1}{\varphi} + 1$$

Therefore, the equality holds:

$$\fint_a^{\varphi\, b} (g(x) + f(x))\,dx = A *^{\varphi} B$$

□

14 Discrete Form of the Linear Mean

Building from the continuous definition and the rigorous restoration of the function-centric identity, the linear mean $\tilde{F}_{\varphi}^{t}(x)$ is expressed as:

$$\begin{aligned} \tilde{F}_{\varphi}^{t}(x) &= \fint_{x-t}^{\varphi\, x} \frac{\ln(\varphi F(x) + 1 - \varphi)}{t\varphi}\,dx \\ &= \fint_{x-t}^{\varphi\, x} \ln\left((\varphi F(x) + 1 - \varphi)^{\frac{1}{t\varphi}}\right) dx \end{aligned}$$

14.1 Derivation of the Discrete Form

To find the discrete counterpart, we discretize the interval into $|N|$ elements and evaluate the resulting product-limit:

1. **Iterative Product Definition:** Through the property of the linear integral as an exponentiated average, we obtain:

$$\begin{aligned} \tilde{F}_{\varphi}^{t}(x) &= \frac{\prod_{i=1}^{|N|} (\varphi F(x_i) + 1 - \varphi)^{\frac{1}{|N|}} - 1}{\varphi} + 1 \\ &= \frac{\left(\prod_{i=1}^{|N|} e^{\ln(\varphi F(x_i) + 1 - \varphi) \cdot \frac{1}{|N|}}\right) - 1}{\varphi} + 1 \end{aligned}$$

2. **Re-expressing in Set Notation:** For a set S defined over the interval $[x-t, x]$, the moving average is calculated relative to the cardinality $|N|$:

$$\tilde{F}_{\varphi}^{t}(x) = \frac{\prod_{x \in S} (\varphi F(x) + 1 - \varphi)^{\frac{1}{|N|}} - 1}{\varphi} + 1$$

14.2 Unified Discrete Linear Mean ($\tilde{S}_{\varphi}$)

For a general set S, where we treat the elements themselves as the function values ($F(x) = x$), we define the **Unified Linear Mean**:

$$\tilde{S}_{\varphi} = \frac{\prod_{\varphi x \in S} (x + 1 - \varphi)^{\frac{1}{|S|}}}{\varphi} + \frac{\varphi - 1}{\varphi} \tag{18}$$

15 Properties and Literature Context

The Unified Linear Mean $\tilde{S}_\varphi$ represents a structural interpolation between the additive and multiplicative regimes. Its properties derive directly from the algebraic structure of the aggregation operator $*^\varphi$ and the functional restoration established in the previous sections.

16 Associativity and the Partition Property

The Unified Linear Mean $\tilde{S}_\varphi$ satisfies the property of **associativity** (decomposability), which allows for the hierarchical calculation of means across partitioned sets. This property is a direct consequence of the coordinate independence established in the functional restoration of $F(x)$.

16.1 Partition Identity

Let a set S be partitioned into two disjoint subsets A and B, such that $|S| = |A| + |B|$. To maintain the structural integrity of the linear integral, the global mean $\tilde{S}_\varphi$ must satisfy:

$$\varphi\tilde{S}_\varphi + 1 - \varphi = \left((\varphi\tilde{A}_\varphi + 1 - \varphi)^{|A|} \cdot (\varphi\tilde{B}_\varphi + 1 - \varphi)^{|B|}\right)^{\frac{1}{|A|+|B|}} \tag{19}$$

16.2 The Linearized State Operator

The elegance of this property is best viewed through the **Linearized State Operator** $\mathbb{L}_\varphi(x) = \varphi x + 1 - \varphi$, which appeared as the core kernel in the rigorous cancellation of F_0. Using this operator, the partition property takes the form of a weighted geometric mean of the linearized states:

$$\mathbb{L}_\varphi(\tilde{S}_\varphi) = \left(\mathbb{L}_\varphi(\tilde{A}_\varphi)^{|A|} \cdot \mathbb{L}_\varphi(\tilde{B}_\varphi)^{|B|}\right)^{\frac{1}{|S|}} \tag{20}$$

16.3 Limit Consistency

This rigorous form ensures that the partition logic remains consistent across the φ spectrum:

- **Geometric Case ($\varphi = 1$):** $\mathbb{L}_1(x) = x$. The identity resolves to the standard geometric mean associativity: $\hat{S} = (\hat{A}^{|A|} \cdot \hat{B}^{|B|})^{1/|S|}$.
- **Arithmetic Case ($\varphi \to 0$):** Through the application of L'Hôpital's rule to the logarithmic form of $\mathbb{L}_\varphi$, the identity resolves to the weighted arithmetic sum: $\bar{S} = \frac{|A|}{|S|}\bar{A} + \frac{|B|}{|S|}\bar{B}$.

16.4 Connections to Existing Literature

The construction of the linear mean aligns with several advanced frameworks in mathematical analysis and statistical mechanics:

- **Quasi-Arithmetic (Kolmogorov-Nagumo) Means:** The linear mean is a realization of the generalized mean $M_f(x) = f^{-1}\left(\frac{1}{n}\sum f(x_i)\right)$. In this framework, the derived generator is $f(x) = \ln(x\varphi + 1 - \varphi)$.
- **Non-Newtonian Calculus:** By providing a continuous transition between arithmetic ($\varphi = 0$) and geometric ($\varphi = 1$) means, this work bridges standard Newtonian calculus and the Multiplicative Calculus of Grossman and Katz.

- **Tsallis Statistics:** The φ parameter functions similarly to the entropic index q in non-extensive thermodynamics, where the standard additive laws are modified to describe systems with long-range interactions.

16.5 Summary of Mean Consistency

As established by the rigorous restoration of the $F(x)$-centric form, the mean satisfies the following essential criteria:

1. **Coordinate Independence:** The value $\tilde{S}_\varphi$ is independent of the global reference x_0.
2. **Reflexivity:** If all elements $x \in S$ are equal to k, then $\tilde{S}_\varphi = k$.
3. **Monotonicity:** The mean is strictly increasing with respect to any individual element x_i for $\varphi \in [0, 1]$.