Randomness and e

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Abstract

Unlike π, which comes directly from the area or circumference of a circle, e seems to be an arbitrary concept disconnected from nature at the first glance. So what relevance does this irrational constant bear besides bank accounts that only compound their interest continuously? It turns out that e can be found in something as simple as picking random numbers between 0 and 1.

The problem

Imagine one is to pick numbers between 0 and 1 until the sum of them gets larger than 1. For example, the first number picked is 0.579, and second 0.234; now their sum 0.813 hasn't exceeded 1, so we pick another: 0.9005. $0.9005 + 0.813 = 1.7135$ is larger than 1, so we stop here. It took us three numbers get a sum larger than 1.

Then, extending this process, one may want to find out how many numbers he needs on average to reach 1. Intuitively, because 0.5 is half way between 0 and 1, and it takes two 0.5 to get to 1, one may guess 2. But surprisingly, the answer is e. After verifying the result on computer, I decided to try to find a mathematical proof for this statement.

Observations

To get a handle on the problem, I started with some hypothetical scenarios.

1. It takes us only one pick to reach 1:
   Only 1 satisfies this, so the probability of such thing happening is 0%.
2. It takes us two picks to reach 1.

   That means $a + b \geq 1$. Then a can be any number, while b can be any number satisfies $b \geq 1 - a$. That means for each number a, b can be any number that is in the range $[1 - a, 1]$, which gives a probability of $\big(1 - (1 - a)\big) = a$. If each $a \in [0,1)$ has a $\frac{1}{|[0,1)|}$ chance of occurrence, then by rule of product for each $a \in [0,1]$, there is an $\frac{a}{|[0,1)|}$ chance that $b \in [0,1]$ together with a satisfies the condition. Then the full probability of a randomly picked pair $a, b \in [0,1]$ summing up to one is

   $$\sum_{a \in [0,1)} \frac{a}{|[0,1)|}$$

   Because a has an even distribution over the range $[0,1]$, we can remap it with indexing $[0,1] = \left\{a_i | i \in [|[0,1]|], a_i = \frac{i}{|[0,1]|}\right\}$, then we have

$$\mathbb{P}(2) = \sum_{i=1}^{|[0,1)|} \frac{a_i}{|[0,1)|} = \int_0^1 x dx = \frac{1}{2}$$

3. It takes us three picks to reach 1.

   Consider numbers $a, b, c \in [0,1]$; again, for any numbers $a, b \in [0,1]$, there must exist c such that $a + b + c \geq 1$. Thus, to find a pair $(a, b, c) \in [0,1]^3$ that suffices the condition, one can first choose an arbitrary a from $[0,1]$, and an arbitrary b. And for the third choice, since $a + b + c \geq 1$, c has to suffice $c \geq 1 - a - b$. This gives c an event space of $[1 - a - b, 1]$. Then given a pair of $a, b \in [0,1]$, the probability that a randomly picked c satisfies the condition $a + b + c \geq 1$ is $\frac{(1-(1-a-b))}{1} = a + b$. Since each $a, b \in [0,1]$ has $\frac{1}{|[0,1]|}$ chance of occurrence, then by rule of product there is a $\frac{a+b}{|[0,1]|^2}$ chance that one ends up picking the particular pair of a, b and a random c that satisfies $a + b + c \geq 1$. With sum rule, we can sum up the probability of all the different combinations of a, b to find the total possibility of a randomly picked pair (a, b), and a randomly picked c satisfying scenario 3:

$$\sum_{i=1}^{|[0,1)|} \sum_{j=1}^{|[0,1)|} \frac{a_i + b_j}{|[0,1)|^2} = \int_0^1 \int_0^1 (x + y) dy\, dx = \int_0^1 (\frac{1}{2} + x) dx = \frac{1}{2} + \frac{1}{2} = 1$$

   But wait, how can this be 1? If the probability for the occurrence of scenario 3 is 1, then shouldn't it always take three picks to reach 1? Of course not. Looking closely, the fact that $a + b$ alone cannot exceed 1 has been ignored. In fact, after the first pick $a \in [0,1]$, in order for $a + b < 1$, b must be less than $1 - a$. Then the range of b is reduced to $[0, 1 - a)$. Taking into account of this, the original formula becomes:

$$\mathbb{P}(3) = \sum_{i=1}^{|[0,1)|} \sum_{j=1}^{|[0,1-a_i)|} \frac{a_i + b_j}{|[0,1]|^2} = \int_0^1 \int_0^{1-x} (x + y) dy\, dx = \int_0^1 \left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{1-x} dx$$

$$= \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right) dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{2}\right) dx = \frac{1}{2} - \frac{x^3}{6}\Big|_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

4. It takes four picks to reach 1.

   Consider numbers $a, b, c, d \in [0,1]$, such that $a + b < 1, a + b + c < 1, a + b + c + d \geq 1$. That is, $\forall (a, b, c, d) \in [0,1]^4$, if (a, b, c, d) satisfies scenario 4, then: $a \in [0,1), b \in [0,1 - a), c \in [0,1 - a - b), d \in [1 - a - b - c, 1]$ holds. With given values of a, b, c, the probability of d satisfying scenario 4 will again be $a + b + c$. And to count the total probability, we again sum up all the respective probability of the valid pairs of (a, b, c) and a random d satisfying scenario 4. That gives:

$$\mathbb{P}(4) = \sum_{i=1}^{|[0,1)|} \sum_{j=1}^{|[0,1-a_i)|} \sum_{k=1}^{|[0,1-a_i-b_j]|} \frac{a_i + b_j + c_k}{|[0,1]|^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z) dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(xz + yz + \frac{z^2}{2} \right) \Big|_{z=0}^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(x - x^2 - 2xy + y - y^2 + \frac{1 - 2x + x^2 - 2y + 2xy + y^2}{2} \right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{1}{2} - \frac{x^2}{2} - \frac{y^2}{2} - xy \right) dy\, dx$$

$$= \int_0^1 \left(\frac{1}{2} y - \frac{x^2 y}{2} - \frac{y^3}{6} - \frac{xy^2}{2} \right) \Big|_{y=0}^{1-x} dx$$

$$= \int_0^1 \left(\frac{1}{2}(1-x) - \frac{x^2(1-x)}{2} - \frac{(1-x)^3}{6} - \frac{x(1-x)^2}{2} \right) dx$$

$$= \int_0^1 \left(\frac{x^3}{6} - \frac{x}{2} + \frac{1}{3} \right) dx = \left(\frac{x^4}{24} - \frac{x^3}{4} + \frac{x}{3} \right) \Big|_0^1 = \frac{1 - 6 + 8}{24} = \frac{1}{8}$$

5. It takes five picks to reach 1.

$$\mathbb{P}(5) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \int_0^{1-x-y-w} (x+y+z+w) dw\, dz\, dy\, dx = \frac{1}{30}$$

   (This is already too complicated to compute by hand, so I used an online integrator to find the answer)

6. It takes six picks to reach 1.

   (At this point not even the online integrator can handle the integration, so I used a numerical simulation instead)

```
a = 0
b = 0
c = 0.0
d = 0.0
import random
for i in range(0,100000000):
    a+=random.random()
```

```
        b+=1
        if(a>=1):
            d+=1
            if(b==6):
                c+=1
            b=0
            a=0

print c/d
```

The approximated result is 0.00697 ..., which is between $\frac{1}{145}$ and $\frac{1}{143}$.

…

n. It takes n picks to reach 1.

As we have observed, for each pick before the n_{th} one, its value summed with all previous picks cannot exceed 1. That gives us the formula

$$\forall m \in [n-1], p_m \in [0, 1 - \sum_{i=1}^{m-1} p_i)$$

And for the last pick, its sum with all previous terms have to be greater or equal to 1, which means:

$$p_n \in [1 - \sum_{i=1}^{n-1} p_i\,, 1]$$

Because of p_n is picked randomly between 0 and 1, the probability of a valid p_n is

$$\frac{\left(1 - (1 - \sum_i^{n-1} p_i)\right)}{1 - 0} = \sum_{i=1}^{n-1} p_i$$

. Now for a specific set of $\{p_i\}_{i \in [n-1]}$, since each p_i is picked from an event space of $[0,1]$, which contains $|[0,1]|$ elements, the set $\{p_i\}_{i \in [n-1]}$ has a chance of $\frac{1}{|[0,1]|^{n-1}}$ to emerge from a random pick. By rule of product, for any specified set $\{p_i\}_{i \in [n-1]}$, there is a chance of $\frac{\sum_i^{n-1} p_i}{|[0,1]|^{n-1}}$ for us to encounter it along with an ending pick p_n that falls in the valid range $[1 - \sum_i^{n-1} p_i\,, 1]$.

If we union all the valid cases different combinations of $\{p_i\}_{i \in [n-1]}$ constitute, we can cover the whole event space. Thus, by summing up their individual probabilities, which automatically excludes the ones with $p_i, i \in [m-1]$ outside their valid range, the total probability of case n. can be obtained.

Denoting the real space of [0,1] with uniform mapping $\left\{a_i | i \in [|[0,1]|], a_i = \frac{i}{|[0,1]|}\right\}$, then we have:

$$\mathbb{P}(n) = \sum_{i_1=1}^{|[0,1)|} \sum_{i_2=1}^{|[0,1-a_{i_1})|} \cdots \sum_{i_{n-1}=1}^{\left|\left[0,1-\sum_{j=1}^{n-2} a_{i_j}\right)\right|} \frac{\sum_{j=1}^{n-1} a_{i_j}}{|[0,1]|^{n-1}}$$

$$= \int_0^1 \int_0^{1-p1} \underbrace{\cdots}_{n-4} \int_0^{1-\sum_{i=1}^{n-2} p_i} \sum_{j=1}^{n-1} p_j \, dp_{n-1} \underbrace{\cdots}_{n-4} dp_2 dp_1$$

Once the probability of all the scenarios are computed, which ranges from "it takes 1 pick to reach 1" to "it takes ∞ picks to reach 1", a weighted average can be computed to find the average count of numbers needed to sum up to 1:

$$\bar{C} = \sum_{n=1}^{\infty} \mathbb{P}(n) * n = \sum_{n=1}^{\infty} \left[\int_0^1 \int_0^{1-p1} \underbrace{\cdots}_{n-4} \int_0^{1-\sum_{i=1}^{n-2} p_i} \sum_{j=1}^{n-1} p_j \, dp_{n-1} \underbrace{\cdots}_{n-4} dp_2 dp_1 \right] * n$$

Calculation

Now one may be wondering, if $\mathbb{P}(5)$ in the previous discussion was already too complicated to be solved by hand, how are we going to find the accurate value for $\mathbb{P}(6)$, let alone $\mathbb{P}(\infty)$?

We can begin by looking at the first few terms of $\mathbb{P}$: $0, \frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}, \frac{1}{145} \sim \frac{1}{143}, \ldots$ After several tries, it turns out that $\mathbb{P}(n) = \frac{n-1}{n!}$ seems to be a fitting guess:

$$\frac{1-1}{1!} = 1; \frac{2-1}{2!} = \frac{1}{2}; \ldots; \frac{5-1}{5!} = \frac{4}{120} = \frac{1}{30}; \frac{6-1}{6!} = \frac{5}{720} = 144$$

With the observed pattern, we can form the hypothesis that:

$$H_0: \forall n \in \mathbb{N}, \mathbb{P}(n) = \int_0^1 \underbrace{\cdots}_{n-3} \int_0^{1-\sum_{i=1}^{n-2} p_i} \sum_{j=1}^{n-1} p_j \, dp_{n-1} \underbrace{\cdots}_{n-3} dp_1 = \frac{n-1}{n!}$$

Although $n \in \mathbb{N}$ here does suggest the use of induction, the actual implementation of the technique would be a challenge. If one assumes that the statement holds for some $k \in \mathbb{N}$, trying to get to $k+1$ would potentially require the rearrangement of the integrals, which cannot happen because of the dependency of variables between the limits, or insert another layer of integration, which cannot be easily done without expanding out each level of integration. Thus, this expression requires further simplification.

Looking at the expression $\frac{n-1}{n!}$, intuition suggested the separation of numerators: $\frac{n}{n!} - \frac{1}{n!}$, which gives $\frac{1}{(n-1)!} - \frac{1}{n!}$. Equivalating the two sides gives:

$$\forall n \in \mathbb{N}, \mathbb{P}(n) = \int_0^1 \underbrace{\cdots}_{n-3} \int_0^{1-\sum_{i=1}^{n-2} p_i} \sum_{j=1}^{n-1} p_j \, dp_{n-1} \underbrace{\cdots}_{n-3} dp_1 = \frac{1}{(n-1)!} - \frac{1}{n!}$$

I then manipulated the left side of the equation to find a pattern that corresponds to that of the right:

$$\int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} \left(1 - \left(1 - \sum_{j=1}^{n-1} p_j\right)\right) dp_{n-1} \dots dp_1 = \frac{1}{(n-1)!} - \frac{1}{n!}$$

$$\Rightarrow \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} dp_{n-1} \dots dp_1 - \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} \left(1 - \sum_{j=1}^{n-1} p_j\right) dp_{n-1} \dots dp_1 = \frac{1}{(n-1)!} - \frac{1}{n!} \text{ (sum rule);}$$

Observe that $1 - \sum_{j=1}^{n-1} p_j$ is equivalent to $\int_0^{1-\sum_{j=1}^{n-1} p_j} dx$, where x can be any variable,

we can substitute $1 - \sum_{j=1}^{k-1} p_j$ with $\int_0^{1-\sum_{j=1}^{k-1} p_j} dp_k$, yielding:

$$\int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} dp_{n-1} \dots dp_1 - \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_1 = \frac{1}{(n-1)!} - \frac{1}{n!} \qquad (1)$$

Now a really elegant pattern emerges. It seems like $\int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} dp_{n-1} \dots dp_1$ naturally

corresponds with the value of $\frac{1}{(\mathrm{n}-1)!}$, and similarly $\int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_1$ with $\frac{1}{n!}$.

If this correspondence is established, the original hypothesis H_0 can be revised into:

$$H_1: \forall n \in \mathbb{N}, \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_1 = \frac{1}{n!}$$

Notice that due to the symmetry on both sides of the minus sign in (1), proving H_1 to be true automatically validates the whole equation 1, and thus H_0. Thus, we can establish the relation: $H_1 \Rightarrow H_0$.

The statement H_1, nevertheless, is still too complicated to form a useful induction hypothesis. Meanwhile, computing the integral directly may yield some interesting regularities:

$$\int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_1 = \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} \left(1 - \sum_{i=1}^{n-1} p_i\right) dp_{n-1} \dots dp_1$$

$$= \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} (1 - \sum_{i=1}^{n-2} p_i) dp_{n-1} \dots dp_1 - \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} p_{n-1} dp_{n-1} \dots dp_1$$

$$= \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-3} p_i} (1 - \sum_{i=1}^{n-2} p_i)(1 - \sum_{i=1}^{n-2} p_i) dp_{n-2} \dots dp_1 - \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-3} p_i} \frac{(1 - \sum_{i=1}^{n-2} p_i)^2}{2} dp_{n-2} \dots dp_1$$

$$= \int_0^1 \dots \int_0^{1-\sum_{i=1}^{n-3} p_i} \frac{(1 - \sum_{i=1}^{n-2} p_i)^2}{2} dp_{n-2} \dots dp_1 = \cdots$$

In the series of computation, the innermost function being integrated were:

$$\left\{1, \left(1 - \sum_{i=1}^{n-1} p_i\right), \frac{(1 - \sum_{i=1}^{n-2} p_i)^2}{2}, \dots\right\}$$

, which is suggestive of the sequence:

$$S = \left\{\frac{1}{a!}\left(1 - \sum_{i=1}^{n-a} p_i\right)^a\right\}_{a \in [n] \cup \{0\}}$$

The elegance of this sequence is that with each k we use, we get directly the innermost function that resulted from k kevels of integrations with minimal calculation. Using this sequence, we can formulate another hypothesis H_2, which states that:

$$\forall n \in \mathbb{N}, \forall a \in [n], \quad \int_0^{1-\sum_{i=1}^{n-a} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-a+1} = \frac{1}{a!}\left(1 - \sum_{i=1}^{n-a} p_i\right)^a$$

And note that when we equivalate the value of a in this expression to n, we obtain:

$$\int_0^{1-\sum_{i=1}^{n-n} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-k+1} = \frac{1}{n!}\left(1 - \sum_{i=1}^{n-n} p_i\right)^n$$

$$\Rightarrow \int_0^{1} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-k+1} = \frac{1}{n!}(1-0)^n = \frac{1}{n!}$$

which states exactly H_1, thus giving us $H_2 \Rightarrow H_1$.

Now H_2, carrying a simple form and sufficient information, seems like a good place to start using induction proof:

Let $n \in \mathbb{N}$ be arbitrary, fix n. Let $I(a)$ be the statement that

$$\int_0^{1-\sum_{i=1}^{n-a} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-a+1} = \frac{1}{a!}\left(1 - \sum_{i=1}^{n-a} p_i\right)^a.$$

To show that $I(a)$ is true for all $a \in [n]$, we first investigate the base case:

$$\int_0^{1-\sum_{i=1}^{n-a} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-a+1} = \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n = 1 - \sum_{i=1}^{n-1} p_i = \frac{1}{1!}\left(1 - \sum_{i=1}^{n-1} p_i\right)^1 = \frac{1}{a!}\left(1 - \sum_{i=1}^{n-a} p_i\right)^a \Rightarrow I(1)$$

Inductive step:

Assume $I(k) \Leftrightarrow \int_0^{1-\sum_{i=1}^{n-k} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-k+1} = \frac{1}{k!}\left(1 - \sum_{i=1}^{n-k} p_i\right)^k$ for some $k \in [n-1]$.

Then $\int_0^{1-\sum_{i=1}^{n-(k+1)} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-(k+1)+1}$

$$= \int_0^{1-\sum_{i=1}^{n-(k+1)} p_i} \int_0^{1-\sum_{i=1}^{n-k} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-k+1}\, dp_{n-k}$$

$$= \int_{0}^{1-\sum_{i=1}^{n-(k+1)} p_i} \frac{1}{k!}\left(1-\sum_{i=1}^{n-k} p_i\right)^{k} dp_{n-k}$$

Because variables $\{p_i\}_{i\in[n]}$ are independent to each other, and this integration concerns only with the increment dp_{n-k}, all variables here besides p_{n-k} can be treated as constants, yielding:

$$\frac{d\left(1-\sum_{i=1}^{n-k} p_i\right)}{dp_{n-k}} = \frac{d\left(1-\sum_{i=1}^{n-k-1} p_i - p_{n-k}\right)}{dp_{n-k}} = -1 \Rightarrow dp_{n-k} = -d\left(1-\sum_{i=1}^{n-k} p_i\right)$$

And substituting it back gives:

$$\int_{0}^{1-\sum_{i=1}^{n-(k+1)} p_i} \dots \int_{0}^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-(k+1)+1}$$

$$= \int_{0}^{1-\sum_{i=1}^{n-(k+1)} p_i} -\frac{1}{k!}\left(1-\sum_{i=1}^{n-k} p_i\right)^{k} d\left(1-\sum_{i=1}^{n-k} p_i\right)$$

$$= -\frac{1}{k! * (k+1)}\left(1-\sum_{i=1}^{n-k} p_i\right)^{k+1}\Bigg|_{p_{n-k}=0}^{1-\sum_{i=1}^{n-(k+1)} p_i}$$

$$= -\frac{1}{(k+1)!}\left(1-\sum_{i=1}^{n-k-1} p_i - p_{n-k}\right)^{k+1}\Bigg|_{p_{n-k}=0}^{1-\sum_{i=1}^{n-(k+1)} p_i}$$

$$= -\frac{1}{(k+1)!}\left(1-\sum_{i=1}^{n-k-1} p_i - \left(1-\sum_{i=1}^{n-(k+1)} p_i\right)\right)^{k+1} - \left(-\frac{1}{(k+1)!}\left(1-\sum_{i=1}^{n-k-1} p_i - 0\right)^{k+1}\right)$$

$$= -\frac{1}{(k+1)!}\left(\left(1-\sum_{i=1}^{n-k-1} p_i\right) - \left(1-\sum_{i=1}^{n-k-1} p_i\right)\right)^{k+1} + \frac{1}{(k+1)!}\left(1-\sum_{i=1}^{n-(k+1)} p_i - 0\right)^{k+1}$$

$$= -\frac{1}{(k+1)!} * 0 + \frac{1}{(k+1)!}\left(1-\sum_{i=1}^{n-(k+1)} p_i - 0\right)^{k+1} = \frac{1}{(k+1)!}\left(1-\sum_{i=1}^{n-(k+1)} p_i - 0\right)^{k+1}.$$

$\Rightarrow I(k+1)$.

Since $I(1)$ is true, and from $I(k), k \in [n-1]$, one can obtain $I(k+1), k+1 \in [n]$, by principle of mathematical induction, we can conclude that:

$$\forall a \in [n], \int_0^{1-\sum_{i=1}^{n-a} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-a+1} = \frac{1}{a!}\left(1 - \sum_{i=1}^{n-a} p_i\right)^a,$$

and since $n \in \mathbb{N}$ is arbitrary, we know

$$H_2 \Leftrightarrow \forall n \in \mathbb{N}, \forall a \in [n], \int_0^{1-\sum_{i=1}^{n-a} p_i} \dots \int_0^{1-\sum_{i=1}^{n-1} p_i} dp_n \dots dp_{n-a+1} = \frac{1}{a!}\left(1 - \sum_{i=1}^{n-a} p_i\right)^a$$

holds, and hence $H_2 \Rightarrow H_1 \Rightarrow H_0 \Rightarrow \mathbb{P}(n) = \int_0^1 \underbrace{\dots}_{n-3} \int_0^{1-\sum_{i=1}^{n-2} p_i} \sum_{j=1}^{n-1} p_j \, dp_{n-1} \underbrace{\dots}_{n-3} dp_1 = \frac{n-1}{n!}$

Now with this pattern shown true, one can acquire the average quantity of random pick with aforementioned formula:

$$\bar{C} = \sum_{n=1}^{\infty} \mathbb{P}(n) * n = \sum_{n=1}^{\infty} \frac{n-1}{n!} * n = \sum_{n=1}^{\infty} \frac{n-1}{(n-1)!} = \frac{1-1}{(1-1)!} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = 0 + \sum_{m=0}^{\infty} \frac{1}{m!} = e,$$

and hence the desired result.

Conclusion

This result, although requiring some complicated deduction to arrive at, shows that e indeed exists in nature at places like probability. And for a side note, when infinite picks are allowed, one will be bound to end up with a sum of 1. That means the sum of the probability $\mathbb{P}$ of all the possible events will be 1. To see that, we have:

$$\sum_{n=1}^{\infty} \mathbb{P}(n) = \sum_{n=1}^{\infty} \frac{n-1}{n!} = \sum_{n=1}^{\infty} \frac{n}{n!} - \sum_{n=1}^{\infty} \frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} - \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$= \frac{1}{(1-1)!} + \left(\sum_{n=2}^{\infty} \frac{1}{(n-1)!} - \sum_{n=1}^{\infty} \frac{1}{n!}\right) = 1,$$

exemplifying the correctness of probability. Besides connecting e to natural events, through this calculation we also see the power of thinking. In terms of computation, numerically solving an integration, which involves summing together infinite terms, demands a complexity of $O(\infty)$, solving a double integral $O(\infty^2)$, and an infinitely nested integral $O(\infty^\infty)$. And for our problem, we are taking a sum of the results of infinitely many infinite integrals, which means the complexity will be approximately $O(\infty * \infty^\infty)$. With this tendency, even if we try to obtain an approximated result from finite computations, the time complexity $O(n^n)$ is still substantial. However, with the creativity and knowledge of the mind, one can tackle a problem with complexity that goes far beyond the capabilities of a computer. And creativity rooted in understanding is something that not even artificial intelligence today possess, and perhaps never will. And, hence, this problem reminds us that, even in this world where almost everything can be looked up online, knowledge will always have its place.